The sum for $n=0$ is obviously ∞ and so is for $n=1$ which is just the ***harmonic series*** which is known to diverge to infinity.

It appears that such sums, where the binomial reciprocals appear in the denominator, are still very much a ***research topic***. The problem at hand has ***several known solutions***. I chose the one that appeals to the ***telescoping property*** of the series involved.

By definition, $\displaystyle\frac{1}{C_k^{n+k}}=\frac{n!k!}{(n+k)!}=\frac{n!}{(k+1)(k+2)\cdots(k+n)}$.

Invoking the concept of ***partial fractions***,

$$\frac{1}{(k+1)(k+n)}=\frac{1}{n-1}\left(\frac{1}{k+1}-\frac{1}{k+n}\right),$$

which allows us to write

$$\frac{n!}{(k+1)(k+2)\cdots(k+n)}=\frac{n!}{n-1}\left(\frac{1}{(k+1)\cdots(k+n-1)}-\frac{1}{(k+2)\cdots(k+n)}\right).$$

It follows that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{C_k^{n+k}} &= \frac{n!}{n-1}\sum_{k=0}^{\infty}\left(\frac{1}{(k+1)\cdots(k+n-1)}-\frac{1}{(k+2)\cdots(k+n)}\right)\\ &=\frac{n!}{n-1}\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)\cdots(k+n-1)}-\sum_{k=0}^{\infty}\frac{1}{(k+2)\cdots(k+n)}\right)\\ &=\frac{n!}{n-1}\left(\sum_{k=0}^{\infty}\frac{1}{(k+1)\cdots(k+n-1)}-\sum_{k=1}^{\infty}\frac{1}{(k+1)\cdots(k+n-1)}\right)\\ &=\frac{n!}{n-1}\left.\frac{1}{(k+1)\cdots(k+n-1)}\right|_{k=0}=\frac{n!}{n-1}\,\frac{1}{(n-1)!}=\frac{n}{n-1}.\end{aligned}$$

Bron: http://www.cut-the-knot.org/arithmetic/algebra/BinomialReciprocalsInPascal.shtml