Published by the **Applied Probability Trust**

A Simple Geometric Proof of Morley's Trisector Theorem

BRIAN STONEBRIDGE

Morley's theorem is one of the most surprising and attractive twentieth century results in plane geometry. Its simplicity is part of its beauty, but could easily lead us to expect an equally simple proof. No known proof shows the desirable properties of being purely geometric, concise, and transparent. The lack of such a proof may be a reason why the result is not more widely known.

We provide a simple geometric proof, which relies only on the angle sums of triangles, and the properties of similar triangles and of tangents to a circle. This elementary approach makes the derivation of the result more easily accessible.

Theorem 1 (Morley's Theorem (1899).) *The points of intersection of adjacent trisectors of any triangle form an equilateral triangle.*

Proof Define α, β, and γ such that the angles of $\Delta \boldsymbol{ABC}$ are 3α, 3β, and 3γ. See figure 1. Then, we have the identity

$$\alpha + \beta + \gamma = \frac{\pi}{3} \quad \text{(the angle sum of the triangle).} \tag{1}$$

Start with an *arbitrary* equilateral triangle, ΔXYZ, as shown in figure 2.

1. Let P, Q, and R be points on the altitudes (produced) of ΔXYZ such that

$$\angle XPY (= \angle XPZ) = \alpha + \frac{\pi}{6}, \qquad \angle YQZ (= \angle YQX) = \beta + \frac{\pi}{6},$$
$$\angle ZRX (= \angle ZRY) = \gamma + \frac{\pi}{6}.$$

(If the theorem were indeed true, these would be the values taken by the corresponding angles. We are free to choose them here, and they give (2).)

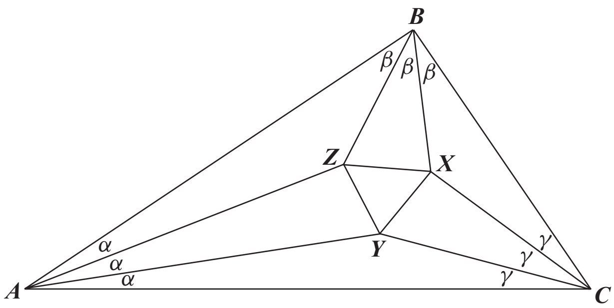

Figure 1 Given any $\Delta \boldsymbol{ABC}$ and its trisectors, we prove that $\Delta \boldsymbol{XYZ}$ is equilateral.

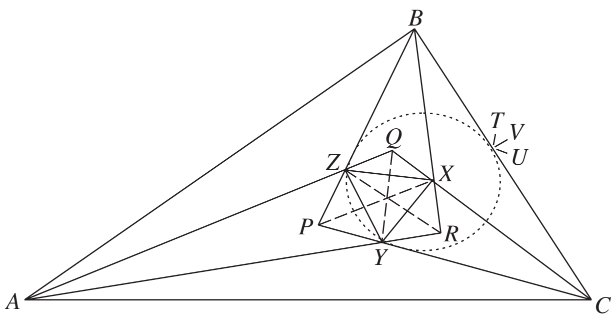

Figure 2 From the equilateral triangle XYZ, construct ΔABC, similar to the given $\Delta \boldsymbol{ABC}$.

2. Let QZ and RY meet at A, let RX and PZ meet at B, and let PY and QX meet at C. Then

$$\angle ZAY = \alpha, \qquad \angle XBZ = \beta, \qquad \angle YCX = \gamma \tag{2}$$

   (by the angle sums of $XRAQ$, $YPBR$, $ZQCP$ (equal to 2π)).

3. Draw a circle with centre X touching PB and, hence, also PC (PX bisects $\angle BPC$).

4. Draw tangents BT and CU, and let them meet at V. Then

$$\angle XBT = \angle XBZ = \beta \quad \text{and} \quad \angle XCU = \angle XCY = \gamma \quad \text{(by (2))}. \tag{3}$$

5. The sum of the angles at P, B, and C in the quadrilateral $PBVC$ is

$$2\alpha + \frac{\pi}{3} + 2\beta + 2\gamma = \pi \quad \text{(by (1))}.$$

   Therefore, $\angle BVC = \pi$ (the angle sum of $PBVC$), and the points T, V, and U coincide.

Therefore, $\angle XBC = \beta$ and $\angle XCB = \gamma$ (by (3)); thus, the angles of ΔXBC are determined.

Similarly, the angles of ΔYCA and ΔZAB are determined by drawing circles with centres at Y and Z.

The above shows that the constructed ΔABC has the same angles as the original $\Delta \boldsymbol{ABC}$, and the trisectors of ΔABC form an equilateral triangle, ΔXYZ. Hence, the same is true of the original $\Delta \boldsymbol{ABC}$, since it is similar to ΔABC.

Previous proofs

One of the shortest proofs of Morley's theorem is that attributed to Penrose (reference 5). Longer proofs are given by Lyness (reference 4), Coxeter (reference 2), and Naraniengar (1909) (which appears in Coxeter and Greitzer (reference 1) and Honsberger (reference 3)). Sastry (reference 6) referred to the latter for a proof, indicating that amongst methods employing simple Euclidean geometry, the method of Naraniengar had not been bettered. This opinion is reinforced in the historical background of the theorem, provided by Guy (reference 7).

We refer the reader to http://www.cut-the-knot.org/triangle/Morley/ for twelve links to a variety of proofs, allowing their merits to be compared.

Acknowledgements

Sincere thanks are due to Jim Gowers, Bill Millar, and John Shepherdson who very patiently moderated the sequence of contortions which led to this article.

References

1 H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited* (MAA, Washington, DC, 1967).
2 H. S. M. Coxeter, *Introduction to Geometry*, 2nd edn. (Wiley, New York, 1969).
3 R. Honsberger, Morley's theorem, in *Mathematical Gems I* (MAA, Washington, DC, 1973), pp. 92–99.
4 R. C. Lyness, Angles, circles and Morley's theorem, in *Mathematical Reflections*, eds Members of the Association of Teachers of Mathematics (Cambridge University Press, 1968).
5 R. Penrose, Morley's trisector theorem, *Eureka* **16** (1953), pp. 6–7.
6 K. R. S. Sastry, Morley's theorem, *Math. Spectrum* **23** (1990/91), p. 1.
7 R. K. Guy, The lighthouse theorem, Morley & Malfatti — a budget of paradoxes, *Amer. Math. Monthly* **114** (2007), pp. 97–141.

Added in proof

The above proof is a 'backwards' proof, its main virtue being its brevity. I am pleased to say that I have subsequently produced a 'forwards' proof, which is also Euclidean, though longer and less transparent. This can be found at http://www.cut-the-knot.org/triangle/Morley/sb.shtml.

Brian Stonebridge *spent over 30 years as a lecturer in Mathematics and Computer Science at the University of Bristol, doing research in optimization, combinatorics, and graph theory. Now, in retirement, his main relaxations are hill walking, choral singing, and tackling dormant mathematical challenges*