PYTHAGORAS AND LINEAR TRANSFORMATIONS

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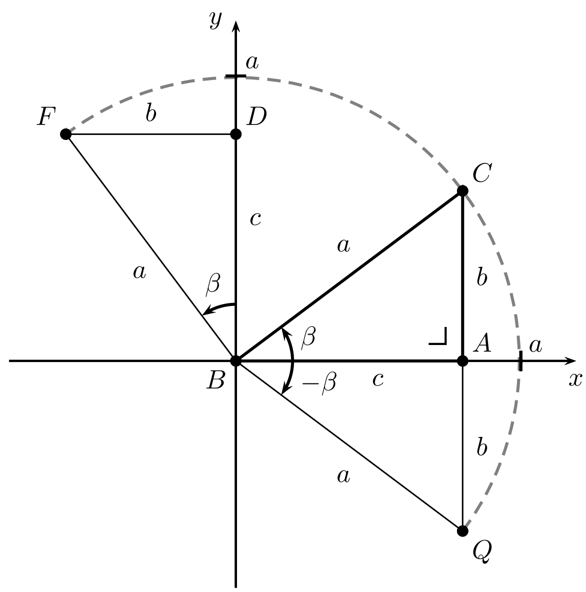
Abstract. In this note we prove the Pythagorean theorem using linear transformations and their matrix representations.

Convention. Throughout, we work in the Euclidean plane. The vertices of a triangle are denoted by big Latin letters, while small Latin letters are used for the sides: Each side will have the same letter as the opposite vertex. The angles are denoted by Greek letters, the letter of an angle will correspond to the letter of the adjacent vertex. A triangle with vertices A, B and C is denoted ΔABC.

Pythagorean theorem. Let ABC be a triangle, with right angle in A. Then $a^2 = b^2 + c^2$ holds.

Proof. We introduce the orthonormal coordinate system as indicated in the picture below. Note that the right angle in A implies that y-axis and line AC are parallel.



Consider the linear[1] transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ associated with the rotation with center the orgin and angle β. Then $T(a, 0) = (c, b)$ and with use of the congruence of triangles ΔDBF and ΔABC one finds $T(0, a) = (-b, c)$. Hence the matrix M associated with the linear transformation T and ordered standard bases $\mathcal{E} = \{(1, 0), (0, 1)\}$ is given by [2]

$$M = \begin{bmatrix} c/a & -b/a \\ b/a & c/a \end{bmatrix}$$

Same reasoning for the linear transformation T^{-1} associated with the rotation with center the orgin and angle $-\beta$, where one replaces traingle ABC by the congruent traingle ABQ, yields the inverse matrix of M

$$M^{-1} = \begin{bmatrix} c/a & b/a \\ -b/a & c/a \end{bmatrix}$$

Finally, working out the matrix identity $M \cdot M^{-1} = I$, we get

$$\begin{bmatrix} c/a & -b/a \\ b/a & c/a \end{bmatrix} \cdot \begin{bmatrix} c/a & b/a \\ -b/a & c/a \end{bmatrix} = \begin{bmatrix} c^2/a^2 + b^2/a^2 & 0 \\ 0 & b^2/a^2 + c^2/a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and comparing the corresponding elements proves $a^2 = b^2 + c^2$. □

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[1]The fact that the map T is linear, i.e. satisfies $\forall u, v \in \mathbb{R}^2 : T(u + v) = T(u) + T(v)$ and $\forall r \in \mathbb{R}, \forall u \in \mathbb{R}^2 : T(ru) = rT(u)$, may be proved by using congruence conditions.

[2]The matrix associated with a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and ordered bases $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$ is determined by placing the coordinates of $T(v_i)$ with respect to the basis $\mathcal{B}$ in the i-th column.