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Total Scalar Curvature of Tubes about Curves

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1. Introduction

Let M be a RIEMANNian manifold of class C^ω and $G_m(s)$ a geodesic sphere with center $m \in M$ and radius s. In [8] A. GRAY and the second author determined a power series expansion for the volume $S_m(s)$ of $G_m(s)$. The main purpose of [8] is to try to characterize EUCLIDean space and the rank one symmetric spaces by means of the volume function $S_m(s)$. For example the authors consider the following conjecture:

Let M be an n-dimensional RIEMANN*ian manifold of class C^ω and suppose that for all $m \in M$ and all sufficiently small geodesic spheres the volume $S_m(s)$ is the same as for* EUCLID*ean space, i.e. $S_m(s) = c_{n-1}s^{n-1}$. Then M is locally flat.* (Here c_{n-1} denotes the volume of the $(n-1)$-dimensional unit sphere in E^n.)

Similar conjectures are given for the rank one symmetric spaces. These questions are answered affirmatively in many important cases but the general problem remains open. In addition B.-Y. CHEN and the second author treated the differential geometry of geodesic spheres extensively in [4] and [5] where a lot of similar problems are studied. In these papers the authors determine mainly properties of $G_m(s)$ considered as a submanifold of the ambient space M.

Next let σ be a topologically embedded curve of finite length $L(\sigma)$ in M and denote by P_s a tube about σ with sufficiently small radius s to avoid focal points of σ. In a subsequent paper [9] a power series expansion is given for the volume $S_\sigma(s)$ of P_s and similar conjectures are studied; for example:

Let M be an n-dimensional RIEMANN*ian manifold of class C^ω and suppose that for all sufficiently short geodesics σ the volume $S_\sigma(s)$ of all sufficiently small tubes P_s about σ is the same as for* EUCLID*ean space, i.e. $S_\sigma(s) = c_{n-2}s^{n-2}L(\sigma)$. Then M is locally flat.*

In contrast with the case of geodesic spheres, the answers are now affirmative in all the cases.

In addition the authors prove in [9] that the volume $S_\sigma(s)$ of a tube about an arbitrary curve in EUCLIDean space or in a rank one symmetric space does not depend on the embedding. It depends in general only on the radius s and the

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length of σ. In this way they generalize a remarkable result of WEYL for tubes about arbitrary submanifolds in E^n or S^n [12] to curves in all rank one symmetric spaces. As proved in [10] this extension does not longer hold for higher dimensional submanifolds, since already for tubes about surfaces in the complex projective space the volume does depend on the embedding.

The technique to attack the volume problem for geodesic spheres is the use of normal coordinates and normal vector fields. For tubes one adapts this and uses FERMI coordinates and FERMI vector fields (see [9], [10]). Further the curvature function of a curve σ and also the torsion operator of σ, as introduced in [9], are used in an extensive way. Although the volume of a tube in general depends on the curvature, it is a remarkable property that it is independent of the torsion of σ. An elegant way to prove this is to develop a method which is a generalization of the so-called LEDGER technique for geodesic spheres. This method is used to study harmonic manifolds (see [2], [11]). The key fact to derive the generalization is a nice relation which exists between FERMI vector fields and JACOBI vector fields. This relation is derived in [9] for curves and in [7] for general submanifolds. After giving some preliminaries in section 2 we develop this new method in section 3. From this we derive at once a power series expansion for the second fundamental form of a tube about σ. It turns out that this form does not depend on the torsion of the curve. The same property is valid for the RIEMANN curvature tensor, the RICCI tensor and the scalar curvature of the tube. This will be proved by using the GAUSS equation in section 4 where we derive power series expansions for these tensors and this function.

We note that the method developed in section 3 provides an alternative way to derive a power series expansion for the volume element and hence the volume of the tube.

In section 5 we determine a power series expansion for the total scalar curvature of a tube about σ. The main purpose is to prove characterization theorems similar to those using the volume function. Here again we have affirmative answers except in the three-dimensional case where the situation is more complicated and related to the GAUSS-BONNET theorem. Also we mention the result that the total scalar curvature for tubes about curves in E^n and the rank one symmetric spaces does not depend on the embedding. This result can be obtained by explicit calculations but those become quickly very complicated. There is another elegant method to prove this fact. This is done in [10] by combining the formulas for the volume of a tube and the STEINER formulas for the volume of parallel hypersurfaces. These formulas are derived in [1] and the integrated mean curvatures appear in a natural way. At the end of this section we give some particular properties for 4-dimensional manifolds.

As stated above it is possible to give the complete power series expansions when σ is a curve in E^n or a rank one symmetric space. In section 6 we show how this can be done for E^n, S^n and H^n.

Finally we note that the method developed in this paper makes it possible to consider other functions for the tubes as for example the total mean curvature, the norm of the second fundamental form, etc. These and several other notions will be considered in [6]. (See also [4], [5].) It is also possible to adapt the method to the case of tubes about arbitrary submanifolds. We refer to [10] where we will prove that also in this situation the torsion of the submanifold disappears in most of the formulas.

2. Preliminaries

Let (M, g) be an n-dimensional RIEMANNian manifold of class C^ω. Denote by $\mathfrak{X}(M)$ the C^∞ vector fields on M, and let ∇ and R be the RIEMANNian connection and the curvature operator of M. They are given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) - g(X,[Y,Z]) \\ &\quad - g(Y,[X,Z]) + g(Z,[X,Y]) , \\ R_{XY} &= \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] , \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Also let ϱ and τ denote the RICCI tensor and the scalar curvature of M.

Let $\sigma : (a, b) \to M$ be a curve in M. To describe the geometry of a RIEMANNian manifold M in the neighborhood of a curve σ we use FERMI *coordinates*. We give now some useful definitions and properties. For an extensive treatment we refer to [9] and to [10] for a generalization to arbitrary submanifolds of a RIEMANNian manifold.

Definition 2.1. *Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame field along a curve $\sigma : (a, b) \to M$ and let $m = \sigma(0)$ be a point on σ. Assume that $\dot\sigma(t) = E_1|_{\sigma(t)}$. Then the* FERMI *coordinates $(x_1, \ldots, x_n)$ of (M, σ) relative to $\{E_1, \ldots, E_n\}$ and m are given by*

$$\begin{aligned} x_1\left(\exp_{\sigma(t)} \sum_{j=2}^{n} t_j E_j|_{\sigma(t)}\right) &= t , \\ x_i\left(\exp_{\sigma(t)} \sum_{j=2}^{n} t_j E_j|_{\sigma(t)}\right) &= t_i , \quad 2 \leqq i \leqq n . \end{aligned}$$

The FERMI coordinates are defined on an open neighborhood $\mathfrak{U}$ of σ. We assume always that $\mathfrak{U}$ contains no focal points of σ.

Definition 2.2. *Let $(x_1, \ldots, x_n)$ be a* FERMI *coordinate system of (M, σ) relative to $(E_1, \ldots, E_n)$ and m. Then $X \in \mathfrak{X}(\mathfrak{U})$ is a* FERMI *vector field for (M, σ) provided $X = \sum_{i=2}^{n} c_i \dfrac{\partial}{\partial x_i}$, where the c_i's are constants.*

In what follows we assume that σ is a unit speed curve and put $\dfrac{\partial}{\partial x_1} = A$. Since σ is in general not a geodesic we have to take in account the *curvature* of the curve. Also we shall need the *torsion operator*.

Definition 2.3. *The curvature of σ is the function $\varkappa$ given by*

$$\varkappa(\sigma(t)) = \| \nabla_A A \| (\sigma(t)) .$$

The torsion operator of σ is the linear operator $\perp$ on the space of FERMI *vector fields given by*

$$\perp X = \text{the vector field } Y = \sum_{i=2}^{n} c_i(t) \frac{\partial}{\partial x_i} \text{ such that}$$

$$Y |_{\sigma(t)} = \text{the normal component of } (\nabla_A X) |_{\sigma(t)} .$$

Further we recall some fundamental properties from [9].

Lemma 2.4. *Let σ be a unit speed curve in M with curvature $\varkappa$ and torsion operator $\perp$. Then*

i) *there exists a unit vector field U along σ such that*

$$g(U, A)(\sigma(t)) = 0 \quad \text{and} \quad (\nabla_A A)|_{\sigma(t)} = \varkappa(\sigma(t))\, U(\sigma(t)) ;$$

ii) *for any two* FERMI *vector fields $X, Y \in \mathfrak{X}(\mathfrak{U})$ we have*

$$g(\perp X, Y)(\sigma(t)) + g(X, \perp Y)(\sigma(t)) = 0 ;$$

iii) *for any* FERMI *vector field $X \in \mathfrak{X}(\mathfrak{U})$ we have*

$$(\nabla_A X)|_{\sigma(t)} = \{ -\varkappa g(X, U) A + \perp X \} |_{\sigma(t)}$$

and

$$[X, A] = 0 .$$

3. The Second Fundamental Form for a Tube about a Curve

The (solid) *tube* of radius r about a curve $\sigma : (a, b) \to M$ is the set $T(\sigma, r) = \{ \exp_{\sigma(t)} x \mid x \in M_{\sigma(t)},\ \|x\| \leq r,\ g(x, \dot\sigma(t)) = 0,\ a < t < b \}$. Here $M_{\sigma(t)}$ denotes the tangent space of M at the point $\sigma(t)$. We assume always that the radius is less than the distance of σ to its nearest focal point. Further we suppose also that $L(\sigma) < \infty$. For small r the set

$$P_r = \{ p \in T(\sigma, r) \mid d(p, \sigma) = r \}$$

is a smooth hypersurface which we also call a tube.

Our main purpose is to study the curvature properties of such a tube. We develop a method which is an immediate extension of that used to treat geodesic spheres in a RIEMANNian manifold (see for example [2], [11]). We will use in an extensive way JACOBI vector fields and their relation with the FERMI vector fields.

Let p be a point on the tube P_r and let $\gamma(s)$ be the geodesic of M containing p and meeting σ orthogonally at $m = \sigma(0)$. We assume that γ is parametrized by its arc length s and that $\gamma(0) = m$. Further let $(x_1, \ldots, x_n)$ be a FERMI coordinate system of (M, σ) relative to $\{E_1, \ldots, E_n\}$ and m. Hence

$$\dot\sigma(t) = \left. \frac{\partial}{\partial x_1} \right|_{\sigma(t)} = E_1(t) = A \,|_{\sigma(t)} ,$$

$$\left. \frac{\partial}{\partial x_i} \right|_{\sigma(t)} = E_i(t), \quad i = 2, \ldots, n$$

and we take the basis $\{E_1, \ldots, E_n\}$ such that

$$\left.\frac{\partial}{\partial x_2}\right|_{\sigma(0)} = E_2(0) = \gamma'(0) \ ;$$

"$\cdot$" denotes the differentiation with respect to t and "$'$" the differentiation with respect to s.

The JACOBI field equation along γ is

$$Y'' + R_{\gamma' Y}\gamma' = 0 \ .$$

We denote by $\{e_1, \ldots, e_n\}$ the orthonormal frame along γ obtained by parallel translation of $\{E_i(0)\}$ along γ and consider the $n-1$ JACOBI vector fields Y_a, $a = 1, 3, \ldots, n$ with the initial conditions

$$\begin{aligned} &Y_1(0) = E_1(0), \quad Y_1'(0) = A'(0) \ , \\ &Y_\alpha(0) = 0, \qquad\quad Y_\alpha'(0) = E_\alpha(0) \ , \end{aligned} \tag{3.1}$$

where $\alpha = 3, \ldots, n$. Further we put

$$Y_a(s) = Be_a \ .$$

This gives rise to the endomorphism-valued function $s \mapsto B(s)$ and the endomorphism-valued equation

$$B'' + R \circ B = 0$$

where R denotes the endomorphism of $M_{\gamma(s)}$ given by

$$R(s)\, x = R_{\gamma' x}\gamma' \ .$$

The main tool for our treatment here is the following lemma which gives a nice relation between the FERMI vector fields and JACOBI vector fields. The proof is given in [9] (see also [7], [10] for generalizations to arbitrary submanifolds).

Lemma 3.1. *The vector fields*

$$\left.\frac{\partial}{\partial x_1}\right|_{\gamma(s)}, \quad \left.\frac{\partial}{\partial x_2}\right|_{\gamma(s)}, \quad s\left.\frac{\partial}{\partial x_3}\right|_{\gamma(s)}, \ldots, s\left.\frac{\partial}{\partial x_n}\right|_{\gamma(s)}$$

are JACOBI *vector fields along γ. Furthermore they satisfy the initial conditions* (3.1).

From this Lemma 3.1 we obtain

Lemma 3.2. *The endomorphism $B(s)$ satisfies the following initial conditions:*

$$B(0) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \quad B'(0) = \left(\begin{array}{c:c} -\varkappa(m)\, g(U, E_2)\,(m) & 0 \\ \hdashline -\perp(m) & I \end{array}\right) \tag{3.2}$$

where $\perp$ denotes the matrix $(\perp_{\alpha 2})$ with $\perp_{\alpha 2} = g\left(\perp \frac{\partial}{\partial x_\alpha}, e_2\right)$, $\alpha = 3, \ldots, n$.

Proof. This follows at once from Lemma 3.1 and Lemma 2.4.

Next let ω denote the volume form of M (defined locally up to sign) and put

$$\omega_{1\cdots n}=\omega\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right).$$

Then we have

$$\det B(s)=s^{n-2}\vartheta(s)=s^{n-2}\omega_{1\cdots n}(\exp_m se_2) \tag{3.3}$$

(see Lemma 5.1 in [9]). Denote by S the shape operator of the tube P_s. Then we have

$$SY_a=Y_a'$$

and hence, using $Y_a=Be_a$:

$$S=B'B^{-1}\,. \tag{3.4}$$

Our aim is to give a power series expansion for S. To derive this series we will use a trick similar to that used by LEDGER to obtain the so-called LEDGER formulas for small geodesic spheres in a RIEMANNian manifold (see [2], [11]). Put

$$C=sB'B^{-1}\,. \tag{3.5}$$

Then we have

$$sC'=-s^2R-C^2+C\,. \tag{3.6}$$

Taking the n-th derivative of this relation and evaluating it at $s=0$ we get

$$(n-1)\,C^{(n)}(0)=-n\,(n-1)\,R^{(n-2)}(0)-\sum_{k=0}^{n}\binom{n}{k}C^{(k)}(0)\,C^{(n-k)}(0),\quad n\in N_0\,. \tag{3.7}$$

Further, using Lemma 3.1 and (3.4), we get the following initial value for C:

$$C(0)=\left(\begin{array}{c|ccc}0 & 0 & \ldots & 0\\ \hline 0 & & & \\ \vdots & & I & \\ 0 & & & \end{array}\right). \tag{3.8}$$

Now (3.5), (3.7) and (3.8) allow at least theoretically to compute the coefficients in the power series expansion

$$C(s)=\sum_{k=0}^{\infty}\frac{1}{k!}\,C^{(k)}(0)\,s^k.$$

We compute $C'(0)$, $C''(0)$, $C'''(0)$ and $C^{(4)}(0)$

First, from (3.7) we find

$$C'(0)\,C(0)+C(0)\,C'(0)=0$$

and so, by (3.8) we get

$$C'(0)=\left(\begin{array}{c|ccc}\lambda & 0 & \ldots & 0\\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array}\right).$$

To determine λ we use (3.5) in the form $CB = sB'$. Take the derivative and evaluate at $s=0$:

$$C'(0)\ B(0) + C(0)\ B'(0) = B'(0)\ .$$

Using (3.2) we get

$$\lambda = -\varkappa(m)\ g(U, E_2)\ (m)\ .$$

The other coefficients follow then at once from (3.7) and the expressions for $C(0)$ and $C'(0)$. We put

$$R_{ijkl} = g\left(R_{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) \quad \text{and} \quad \nabla^2_{ij} = \nabla^2_{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}}$$

for $i, j, k, l = 1, 2, \ldots, n$. Then we have (note that C is symmetric):

(i) $C''_{11}(0) = -2\ (R_{1212} + \varkappa^2 g(U, E_2)^2)\ (m)$,

$C''_{1\alpha}(0) = -R_{12\alpha 2}(m)$,

$C''_{\alpha\beta}(0) = -\frac{2}{3}\ R_{\alpha 2\beta 2}(m)$;

(ii) $C'''_{11}(0) = -3\ (\nabla_2 R_{1212} + 2R_{1212}\varkappa g(U, E_2) + 2\varkappa^3 g(U, E_2)^3)\ (m)$,

$C'''_{1\alpha}(0) = -(2\nabla_2 R_{12\alpha 2} + R_{12\alpha 2}\varkappa g(U, E_2))\ (m)$,

$C'''_{\alpha\beta}(0) = -\frac{3}{2}\nabla_2 R_{\alpha 2\beta 2}(m)$;

(iii) $$C^{(4)}_{11}(0) = -4\left(\nabla^2_{22}R_{1212} + 2R^2_{1212} + \frac{1}{2}\sum_{\delta=3}^{n} R^2_{12\delta 2} + 2\nabla_2 R_{1212}\varkappa g(U, E_2) + 8R_{1212}\varkappa^2 g(U, E_2)^2 + 6\varkappa^4 g(U, E_2)^4\right)(m)\ ,$$

$$C^{(4)}_{1\alpha}(0) = -\left(3\nabla^2_{22}R_{12\alpha 2} + 3R_{1212}R_{12\alpha 2} + \sum_{\delta=3}^{n} R_{12\delta 2}R_{\alpha 2\delta 2} + 2\nabla_2 R_{12\alpha 2}\varkappa g(U, E_2) + 4R_{12\alpha 2}\varkappa^2 g(U, E_2)^2\right)(m)\ ,$$

$$C^{(4)}_{\alpha\beta}(0) = -\left(\frac{12}{5}\nabla^2_{22}R_{2\alpha 2\beta} + \frac{6}{5}R_{12\alpha 2}R_{12\beta 2} + \frac{8}{15}\sum_{\delta=3}^{n} R_{\alpha 2\delta 2}R_{\beta 2\delta 2}\right)(m)\ ,$$

where $\alpha, \beta = 3, \ldots, n$.

An important consequence of these formulas is that C does not depend on the torsion operator $\perp$.

Now we formulate

Theorem 3.3. *Let σ be a unit speed curve in M and $p = \exp_m su$ a point of the tube P_s about σ. Then the second fundamental form S of P_s at p is given by the following expansions:*

$$S_{11}(p) = -\Big\{\varkappa g(U, u) + s\ (R_{1u1u} + \varkappa^2 g(U, u)^2) + \frac{s^2}{2}\ (\nabla_u R_{1u1u} + 2R_{1u1u}\varkappa g(U, u) + 2\varkappa^3 g(U, u)^3) \tag{3.9}$$

$$+\frac{s^3}{12}\Big(2\nabla^2_{uu}R_{1u1u}+4R^2_{1u1u}+\sum_{\delta=3}^{n}R^2_{1u\delta u}$$

$$+4\nabla_u R_{1u1u}\varkappa g(U,u)+16R_{1u1u}\varkappa^2 g(U,u)^2$$

$$+12\varkappa^4 g(U,u)^4\Big)+0(s^4)\Big\}(m)\ ;$$

$$S_{1\alpha}(p)=-\Big\{\frac{s}{2}R_{1u\alpha u}+\frac{s^2}{6}(2\nabla_u R_{1u\alpha u}+R_{1u\alpha u}\varkappa g(U,u))$$

$$+\frac{s^3}{24}\Big(3\nabla^2_{uu}R_{1u\alpha u}+3R_{1u1u}R_{1u\alpha u}+\sum_{\delta=3}^{n}R_{1u\delta u}R_{\alpha u\delta u}$$

$$+2\nabla_u R_{1u\alpha u}\varkappa g(U,u)+4R_{1u\alpha u}\varkappa^2 g(U,u)^2\Big)+0(s^4)\Big\}(m)\ ;$$

$$S_{\alpha\beta}(p)=\Big\{\frac{1}{s}\delta_{\alpha\beta}-\frac{s}{3}R_{\alpha u\beta u}-\frac{s^2}{4}\nabla_u R_{\alpha u\beta u}$$

$$-\frac{s^3}{180}\Big(18\nabla^2_{uu}R_{\alpha u\beta u}+9R_{1u\alpha u}R_{1u\beta u}+4\sum_{\delta=3}^{n}R_{\alpha u\delta u}R_{\beta u\delta u}\Big)+0(s^4)\Big\}(m)\ ,$$

where $\alpha,\beta=3,\dots,n$ *and* $u=E_2(0)$. *Further, the second fundamental form does not depend on the torsion of* σ.

We conclude this section with some additional remarks. For the *mean curvature* h of the tube P_s we have from (3.3), (3.4) and (3.5) since $(\det B(s))'/\det B(s)=\mathrm{tr}(B'B^{-1})(s)$:

$$h(s)=\frac{1}{s}\,\mathrm{trace}\,C(s)=\frac{n-2}{s}+\frac{\vartheta'(s)}{\vartheta(s)}\ . \tag{3.10}$$

From this formula and the power series expansion for C we can compute power series expansions for the mean curvature h and the volume form ω. This last series is given in [9]. Our method provides an alternative way to obtain this series. Since we shall need it, we write down the expression:

Theorem 3.4. *We have at* $p=\exp_m su$:

$$\omega_{1\cdots n}(p)=1-s\varkappa(m)\,g(U,u)\,(m)-\frac{s^2}{6}(\varrho_{uu}+2R_{1u1u})(m) \tag{3.11}$$

$$-\frac{s^3}{12}(\nabla_u\varrho_{uu}+\nabla_u R_{1u1u}-2\varrho_{uu}\varkappa g(U,u))(m)$$

$$-\frac{s^4}{360}\Big(9\nabla^2_{uu}\varrho_{uu}-5\varrho^2_{uu}+6\nabla^2_{uu}R_{1u1u}-20\varrho_{uu}R_{1u1u}$$

$$+10R^2_{1u1u}+12\sum_{\alpha=3}^{n}R^2_{1u\alpha u}+2\sum_{\alpha,\beta=3}^{n}R^2_{\alpha u\beta u}$$

$$-30\nabla_u\varrho_{uu}\varkappa g(U,u)\Big)(m)+0(s^5)\ .$$

The properties of the mean curvature will be discussed in [6] (see also [4], [5]). Further we have

Theorem 3.5. *The volume form (and hence the volume) and the mean curvature of a tube about a curve do not depend on the torsion of the curve.*

4. Curvature of Tubes about Curves

In this section we compute power series expansions for the RIEMANN curvature tensor, the RICCI tensor and the scalar curvature of a tube P_s about σ. The calculations are much more complicated than those for the geodesic spheres as derived in [8]. This is due to the special direction defined by the tangent vector of the curve σ which leads to the consideration of different types of components. We shall need two facts which we state first.

Lemma 4.1. *Let $\{e_i,\ i=1, \ldots, n\}$ be a parallel frame field along γ. Then we have at $p=\gamma(s)=\exp_m su$:*

$$\text{(4.1)}\qquad \begin{cases} R_{ijkl}(p)=\left\{R_{ijkl}+s\nabla_u R_{ijkl}+\frac{s^2}{2}\nabla^2_{uu}R_{ijkl}+\ldots\right\}(m)\,,\\ \varrho_{ij}(p)=\left\{\varrho_{ij}+s\nabla_u\varrho_{ij}+\frac{s^2}{2}\nabla^2_{uu}\varrho_{ij}+\ldots\right\}(m)\,,\\ \tau(p)=\left\{\tau+s\nabla_u\tau+\frac{s^2}{2}\nabla^2_{uu}\tau+\ldots\right\}(m)\,. \end{cases}$$

The second formula we need is the well-known GAUSS *equation* which relates the RIEMANN curvature tensor R of M with the RIEMANN curvature tensor R^T of the tube P_s. We have

$$\text{(4.2)}\qquad \begin{aligned} R^T(X, Y, Z, W)=R(X, Y, Z, W)&+g(SX, Z)\,g(SY, W)\\ &-g(SX, W)\,g(SY, Z) \end{aligned}$$

where $X, Y, Z, W\in\mathfrak{X}(P_s)$.

From (3.9), (4.1) and (4.2) we obtain

Theorem 4.2. *Let $p=\exp_m su$ be a point of a small tube P_s about σ. The curvature tensor of P_s does not depend on the torsion of σ and is given by*

$$\text{(4.3)}\qquad \begin{aligned} R^T_{1\alpha 1\beta}(p)=\Bigg\{&-\frac{1}{s}\varkappa g(U, u)\,\delta_{\alpha\beta}+\left(R_{1\alpha 1\beta}-R_{1u1u}\delta_{\alpha\beta}-\varkappa^2 g(U, u)^2\,\delta_{\alpha\beta}\right)\\ &+s\Bigg(\nabla_u R_{1\alpha 1\beta}-\frac{1}{2}\nabla_u R_{1u1u}\delta_{\alpha\beta}+\frac{1}{3}R_{\alpha u\beta u}\varkappa g(U, u)\\ &-R_{1u1u}\varkappa g(U, u)\,\delta_{\alpha\beta}-\varkappa^3 g(U, u)^3\,\delta_{\alpha\beta}\Bigg)\\ &+s^2\Bigg(\frac{1}{2}\nabla^2_{uu}R_{1\alpha 1\beta}+\frac{1}{3}R_{1u1u}R_{\alpha u\beta u}-\frac{1}{4}R_{1u\alpha u}R_{1u\beta u} \end{aligned}$$

$$-\frac{1}{6}\nabla^2_{uu}R_{1u1u}\delta_{\alpha\beta}-\frac{1}{3}R^2_{1u1u}\delta_{\alpha\beta}-\frac{1}{12}\sum_{\lambda=3}^{n}R^2_{1u\lambda u}\delta_{\alpha\beta}$$

$$+\frac{1}{4}\nabla_u R_{\alpha u\beta u}\varkappa g(U,u)-\frac{1}{3}\nabla_u R_{1u1u}\varkappa g(U,u)\,\delta_{\alpha\beta}$$

$$+\frac{1}{3}R_{\alpha u\beta u}\varkappa^2 g(U,u)^2-\frac{4}{3}R_{1u1u}\varkappa^2 g(U,u)^2\,\delta_{\alpha\beta}$$

$$\left.\left.-\varkappa^4 g(U,u)^4\,\delta_{\alpha\beta}\right)+0(s^3)\right\}(m)\;;$$

$$R^T_{1\alpha\beta\gamma}(p)=\left\{R_{1\alpha\beta\gamma}-\frac{1}{2}R_{1u\beta u}\delta_{\alpha\gamma}+\frac{1}{2}R_{1u\gamma u}\delta_{\alpha\beta}\right.$$

$$+s\left(\nabla_u R_{1\alpha\beta\gamma}-\frac{1}{3}\nabla_u R_{1u\beta u}\delta_{\alpha\gamma}+\frac{1}{3}\nabla_u R_{1u\gamma u}\delta_{\alpha\beta}\right.$$

$$\left.-\frac{1}{6}R_{1u\beta u}\varkappa g(U,u)\,\delta_{\alpha\gamma}+\frac{1}{6}R_{1u\gamma u}\varkappa g(U,u)\,\delta_{\alpha\beta}\right)$$

$$+s^2\left(\frac{1}{2}\nabla^2_{uu}R_{1\alpha\beta\gamma}+\frac{1}{6}R_{1u\beta u}R_{\alpha u\gamma u}-\frac{1}{6}R_{1u\gamma u}R_{\alpha u\beta u}\right.$$

$$-\frac{1}{8}\nabla^2_{uu}R_{1u\beta u}\delta_{\alpha\gamma}+\frac{1}{8}\nabla^2_{uu}R_{1u\gamma u}\delta_{\alpha\beta}-\frac{1}{8}R_{1u1u}R_{1u\beta u}\delta_{\alpha\gamma}$$

$$+\frac{1}{8}R_{1u1u}R_{1u\gamma u}\delta_{\alpha\beta}-\frac{1}{24}\sum_{\lambda=3}^{n}R_{1u\lambda u}R_{\beta u\lambda u}\delta_{\alpha\gamma}$$

$$+\frac{1}{24}\sum_{\lambda=3}^{n}R_{1u\lambda u}R_{\gamma u\lambda u}\delta_{\alpha\beta}-\frac{1}{12}\nabla_u R_{1u\beta u}\varkappa g(U,u)\,\delta_{\alpha\gamma}$$

$$+\frac{1}{12}\nabla_u R_{1u\gamma u}\varkappa g(U,u)\,\delta_{\alpha\beta}-\frac{1}{6}R_{1u\beta u}\varkappa^2 g(U,u)^2\,\delta_{\alpha\gamma}$$

$$\left.\left.+\frac{1}{6}R_{1u\gamma u}\varkappa^2 g(U,u)^2\,\delta_{\alpha\beta}\right)+0(s^3)\right\}(m)\;;$$

$$R^T_{\alpha\beta\gamma\delta}(p)=\left\{\frac{1}{s^2}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})+R_{\alpha\beta\gamma\delta}-\frac{1}{3}(R_{\beta u\delta u}\delta_{\alpha\gamma}-R_{\beta u\gamma u}\delta_{\alpha\delta}\right.$$

$$+R_{\alpha u\gamma u}\delta_{\beta\delta}-R_{\alpha u\delta u}\delta_{\beta\gamma})+s\left(\nabla_u R_{\alpha\beta\gamma\delta}-\frac{1}{4}R_{\beta u\delta u}\delta_{\alpha\gamma}\right.$$

$$\left.+\frac{1}{4}R_{\alpha u\delta u}\delta_{\beta\gamma}-\frac{1}{4}\nabla_u R_{\alpha u\gamma u}\delta_{\beta\delta}+\frac{1}{4}\nabla_u R_{\beta u\gamma u}\delta_{\alpha\delta}\right)$$

$$+s^2\left(\frac{1}{2}\nabla^2_{uu}R_{\alpha\beta\gamma\delta}+\frac{1}{9}R_{\alpha u\gamma u}R_{\beta u\delta u}-\frac{1}{9}R_{\alpha u\delta u}R_{\beta u\gamma u}\right.$$

$$-\frac{1}{10}\nabla^2_{uu}R_{\beta u\delta u}\delta_{\alpha\gamma}+\frac{1}{10}\nabla^2_{uu}R_{\beta u\gamma u}\delta_{\alpha\delta}-\frac{1}{10}\nabla^2_{uu}R_{\alpha u\gamma u}\delta_{\beta\delta}$$

$$+\frac{1}{10}\nabla^2_{uu}R_{\alpha u\delta u}\delta_{\beta\gamma}-\frac{1}{20}R_{1u\beta u}R_{1u\delta u}\delta_{\alpha\gamma}$$

$$+\frac{1}{20}R_{1u\beta u}R_{1u\gamma u}\delta_{\alpha\delta}-\frac{1}{20}R_{1u\alpha u}R_{1u\gamma u}\delta_{\beta\delta}+\frac{1}{20}R_{1u\alpha u}R_{1u\delta u}\delta_{\beta\gamma}$$

$$-\frac{1}{45}\sum_{\lambda=3}^{n}R_{\beta u\lambda u}R_{\delta u\lambda u}\delta_{\alpha\gamma}+\frac{1}{45}\sum_{\lambda=3}^{n}R_{\beta u\lambda u}R_{\gamma u\lambda u}\delta_{\alpha\delta}$$

$$\left.\left.-\frac{1}{45}\sum_{\lambda=3}^{n}R_{\alpha u\lambda u}R_{\gamma u\lambda u}\delta_{\beta\delta}+\frac{1}{45}\sum_{\lambda=3}^{n}R_{\alpha u\lambda u}R_{\delta u\lambda u}\delta_{\beta\gamma}\right)+0(s^3)\right\}(m),$$

where $\alpha, \beta, \gamma, \delta=3, \ldots, n$.

A first contraction gives the Ricci tensor ϱ^T of the tube P_s.

Theorem 4.3. *Let* $p=\exp_m su$ *be a point of a small tube* P_s *about* σ. *The* Ricci *tensor of* P_s *does not depend on the torsion of* σ *and is given by*

$$\tag{4.4} \varrho^T_{11}(p)=\left\{-\frac{n-2}{s}\varkappa g(U,u)+\varrho_{11}-(n-1)R_{1u1u}-(n-2)\varkappa^2 g(U,u)^2\right.$$

$$+s\left(\nabla_u\varrho_{11}-\frac{n}{2}\nabla_u R_{1u1u}-\frac{3n-5}{3}R_{1u1u}\varkappa g(U,u)\right.$$

$$\left.+\frac{1}{3}\varrho_{uu}\varkappa g(U,u)-(n-2)\varkappa^3 g(U,u)^3\right)$$

$$+s^2\left(\frac{1}{2}\nabla^2_{uu}\varrho_{11}-\frac{n+1}{6}\nabla^2_{uu}R_{1u1u}+\frac{1}{3}R_{1u1u}\varrho_{uu}-\frac{n-1}{3}R^2_{1u1u}\right.$$

$$-\frac{n+1}{12}\sum_{\lambda=3}^{n}R^2_{1u\lambda u}+\frac{1}{4}\nabla_u\varrho_{uu}\varkappa g(U,u)-\frac{4n-5}{12}\nabla_u R_{1u1u}\varkappa g(U,u)$$

$$+\frac{1}{3}\varrho_{uu}\varkappa^2 g(U,u)^2-\frac{4n-7}{3}R_{1u1u}\varkappa^2 g(U,u)^2$$

$$\left.-(n-2)\varkappa^4 g(U,u)^4+0(s^3)\right\}(m);$$

$$\varrho^T_{1\alpha}(p)=\left\{\varrho_{1\alpha}-\frac{n-1}{2}R_{1u\alpha u}+s\left(\nabla_u\varrho_{1\alpha}-\frac{n}{3}\nabla_u R_{1u\alpha u}-\frac{n-3}{6}R_{1u\alpha u}\varkappa g(U,u)\right)\right.$$

$$+s^2\left(\frac{1}{2}\nabla^2_{uu}\varrho_{1\alpha}-\frac{n+1}{8}\nabla^2_{uu}R_{1u\alpha u}+\frac{1}{6}\varrho_{uu}R_{1u\alpha u}-\frac{3n-5}{24}R_{1u1u}R_{1u\alpha u}\right.$$

$$-\frac{n+1}{24}\sum_{\lambda=3}^{n}R_{1u\lambda u}R_{\alpha u\lambda u}-\frac{n-3}{12}\nabla_u R_{1u\alpha u}\varkappa g(U,u)$$

$$\left.-\frac{n-3}{6}R_{1u\alpha u}\varkappa^2 g(U,u)^2+0(s^3)\right\}(m);$$

$$\varrho^T_{\alpha\beta}(p)=\Bigg\{\frac{n-3}{s^2}\,\delta_{\alpha\beta}-\frac{1}{s}\,\varkappa g(U,u)\,\delta_{\alpha\beta}+\varrho_{\alpha\beta}-\frac{n-1}{3}\,R_{\alpha u\beta u}$$
$$-\frac{1}{3}\,\varrho_{uu}\delta_{\alpha\beta}-\frac{2}{3}\,R_{1u1u}\delta_{\alpha\beta}-\varkappa^2 g(U,u)^2\,\delta_{\alpha\beta}$$
$$+s\Bigg(\nabla_u\varrho_{\alpha\beta}-\frac{n}{4}\,\nabla_u R_{\alpha u\beta u}-\frac{1}{4}\,\nabla_u\varrho_{uu}\,\delta_{\alpha\beta}-\frac{1}{4}\,\nabla_u R_{1u1u}\delta_{\alpha\beta}$$
$$+\frac{1}{3}\,R_{\alpha u\beta u}\varkappa g(U,u)-R_{1u1u}\varkappa g(U,u)\,\delta_{\alpha\beta}-\varkappa^3 g(U,u)^3\,\delta_{\alpha\beta}\Bigg)$$
$$+s^2\Bigg(\frac{1}{2}\,\nabla^2_{uu}\varrho_{\alpha\beta}-\frac{n+1}{10}\,\nabla^2_{uu}R_{\alpha u\beta u}+\frac{1}{9}\,\varrho_{uu}R_{\alpha u\beta u}$$
$$+\frac{2}{9}\,R_{1u1u}R_{\alpha u\beta u}-\frac{n+1}{20}\,R_{1u\alpha u}R_{1u\beta u}-\frac{n+1}{45}\sum_{\lambda=3}^{n}R_{\alpha u\lambda u}R_{\beta u\lambda u}$$
$$-\frac{1}{10}\,\nabla^2_{uu}\varrho_{uu}\delta_{\alpha\beta}-\frac{1}{15}\,\nabla^2_{uu}R_{1u1u}\delta_{\alpha\beta}-\frac{1}{3}\,R^2_{1u1u}\delta_{\alpha\beta}$$
$$-\frac{2}{15}\sum_{\lambda=3}^{n}R^2_{1u\lambda u}\delta_{\alpha\beta}-\frac{1}{45}\sum_{\lambda,\mu=3}^{n}R^2_{\lambda u\mu u}\delta_{\alpha\beta}+\frac{1}{4}\,\nabla_u R_{\alpha u\beta u}\varkappa g(U,u)$$
$$-\frac{1}{3}\,\nabla_u R_{1u1u}\varkappa g(U,u)\,\delta_{\alpha\beta}+\frac{1}{3}\,R_{\alpha u\beta u}\varkappa^2 g(U,u)^2$$
$$-\frac{4}{3}\,R_{1u1u}\varkappa^2 g(U,u)^2\,\delta_{\alpha\beta}-\varkappa^4 g(U,u)^4\delta_{\alpha\beta}\Bigg)+0(s^3)\Bigg\}\,(m)\,,$$

where $\alpha,\beta=3,\ldots,n$.

In the next section we will write down the complete expansion for the Ricci tensor of a tube about a curve in a space of constant sectional curvature. It will turn out that all the tubes are η-Einstein *hypersurfaces*, i.e. the Ricci tensor has only two eigenvalues, one with multiplicity $n-2$ and one with multiplicity 1. This last one corresponds to the vector e_1. Now we will prove the converse.

Theorem 4.4. *Let M be an n-dimensional* Riemann*ian manifold with $n>3$. Assume that any sufficiently small tube about an arbitrary geodesic σ in M is an η-*Einstein *manifold with respect to the parallel displaced tangent vector of σ. Then M has constant sectional curvature.*

Proof. P_s is η-Einsteinian with the distinguished direction determined by e_1 if and only if

$$\varrho^T=\lambda g^T+\mu\eta\otimes\eta$$

where $\eta(X)=g^T(X,e_1)$, $X\in\mathfrak{X}(P_s)$. This condition implies with (4.4)

$$\varrho_{\alpha\alpha}-\frac{n-1}{3}\,R_{\alpha u\alpha u}=\varrho_{\beta\beta}-\frac{n-1}{3}\,R_{\beta u\beta u}\,.$$

Since the hypothesis is true for any tube we may interchange β and 1. Hence

$$\varrho_{\alpha\alpha}-\frac{n-1}{3}R_{\alpha u\alpha u}=\varrho_{11}-\frac{n-1}{3}R_{1u1u}\,.$$

Now we sum with respect to α and get

$$\tau-\frac{n+2}{3}\varrho_{uu}=(n-1)\,\varrho_{11}-\frac{(n-1)^2}{3}R_{1u1u}\,.$$

Nex we change the role of u and 1. So we get at once that $\varrho_{uu}=\varrho_{11}$ and since u and 1 are arbitrary the manifold must be EINSTEINian. Then (4.4) implies

$$R_{ikjk}=0$$

for any triplet of orthonormal vectors $E_i(0)$, $E_j(0)$, $E_k(0)$ at m. So CARTAN's lemma [3] gives the required result.

Finally we derive the power series expansion for the scalar curvature τ^T of P_s. This follows immediately from Theorem 4.3 by contraction.

Theorem 4.5. *Let $p=\exp_m su$ be a point of a small tube P_s about σ. The scalar curvature τ^T of P_s does not depend on the torsion of σ and is given by*

$$\begin{aligned}(4.5)\qquad \tau^T(p)=\Bigg\{&\frac{(n-3)\,(n-2)}{s^2}-\frac{2\,(n-2)}{s}\varkappa g(U,u)+\tau-\frac{2n}{3}\varrho_{uu}-\frac{4n-6}{3}R_{1u1u}\\ &-2\,(n-2)\,\varkappa^2 g(U,u)^2+s\Bigg(\nabla_u\tau-\frac{n+1}{2}\nabla_u\varrho_{uu}-\frac{n-1}{2}\nabla_u R_{1u1u}\\ &+\frac{2}{3}\varrho_{uu}\varkappa g(U,u)-\frac{6n-10}{3}R_{1u1u}\varkappa g(U,u)-2\,(n-2)\,\varkappa^3 g(U,u)^3\Bigg)\\ &+s^2\Bigg(\frac{1}{2}\nabla^2_{uu}\tau-\frac{n+2}{5}\nabla^2_{uu}\varrho_{uu}-\frac{2n-1}{15}\nabla^2_{uu}R_{1u1u}+\frac{1}{9}\varrho^2_{uu}\\ &+\frac{4}{9}R_{1u1u}\varrho_{uu}-\frac{6n-7}{9}R^2_{1u1u}-\frac{4n-2}{15}\sum_{\lambda=3}^{n}R^2_{1u\lambda u}\\ &-\frac{2n-1}{45}\sum_{\lambda,\mu=3}^{n}R^2_{\lambda u\mu u}+\frac{1}{2}\nabla_u\varrho_{uu}\varkappa g(U,u)-\frac{4n-5}{6}\nabla_u R_{1u1u}\varkappa g(U,u)\\ &+\frac{2}{3}\varrho_{uu}\varkappa^2 g(U,u)^2-\frac{8n-14}{3}R_{1u1u}\varkappa^2 g(U,u)^2\\ &-2\,(n-2)\,\varkappa^4 g(U,u)^4\Bigg)+0(s^3)\Bigg\}\,(m)\,.\end{aligned}$$

In [5] we have considered RIEMANNian manifolds such that the scalar curvature of a small geodesic sphere is constant on the geodesic sphere and proved that this property characterizes harmonic manifolds.

As in [9] we consider now a kind of harmonicity with respect to geodesics.

Definition 4.6. *A* Riemann*ian manifold is said to be scalar curvature harmonic with respect to geodesics provided that for any geodesic σ the scalar curvature τ^T of any small tube P_s about σ depends only on the radius s.*

Just as is [9] we can determine *all* the manifolds which are scalar curvature harmonic with respect to geodesics.

Theorem 4.7. *A* Riemiann*an manifold M is scalar curvature harmonic with respect to geodesics if and only if M has constant curvature.*

Proof. That spaces of constant curvature are scalar curvature harmonic with respect to geodesics will follow from equation (6.1).

Conversely, suppose M is scalar curvature harmonic with respect to geodesics. Then (4.5) implies

$$\tau-\frac{2n}{3}\varrho_{uu}-\frac{4n-6}{3}R_{1u1u}=\alpha \tag{4.6}$$

where α does not depend on u and 1. This implies at once that M is Einstein*ian* (change the role of u and 1). Then (4.6) implies that M has constant sectional curvature.

5. Total Scalar Curvature of Tubes

Again let σ be a sufficiently short unit speed curve in M. Then the *total scalar curvature* $T_\sigma(s)$ of the tube P_s is given by

$$T_\sigma(s)=\int_{P(s)}\tau^T(p)\,dp\,. \tag{5.1}$$

Using the volume form ω, (5.1) becomes

$$T_\sigma(s)=s^{n-2}\int_a^b\int_{S^{n-2}(1)}(\tau^T\omega_{1\cdots n})\,(\exp_{\sigma(t)}su)\,du\,dt$$

where du is the volume element of the $(n-2)$-dimensional unit sphere in E^{n-1}.

Just as for the volume $S_\sigma(s)$ of P_s in [9], we will give a power series expansion for $T_\sigma(s)$. Therefore we use exactly the same method as used in [8], [9], [10]. We delete the details. We have

Theorem 5.1. *The total scalar curvature $T_\sigma(s)$ of a small tube P_s about a curve σ in M is given by*

$$T_\sigma(s)=c_{n-2}s^{n-4}\int_a^b\{(n-3)\,(n-2)+A(n)\,s^2+B(n)\,s^4+0(s^6)\}\,dt \tag{5.2}$$

where

$$A(n)=-\frac{n-3}{6\,(n-1)}\{(n-4)\,\tau+(n+2)\,\varrho_{11}\}\,(\sigma(t))\,,$$

$$
\begin{aligned}
B(n)=\frac{1}{n^2-1}\Big\{&\frac{n^2-9n+2}{72}\tau^2+\frac{n^2+3n+17}{45}\|\varrho\|^2-\frac{(n+1)(n+2)}{120}\|R\|^2\\
&-\frac{(n-3)(n-4)}{20}\Delta\tau-\frac{(n+6)(n-3)}{40}\Delta\varrho_{11}+\frac{11n^2-27n+142}{120}\nabla^2_{11}\tau\\
&+\frac{(n-4)(n+1)}{36}\tau\varrho_{11}+\frac{7n^2+21n-46}{180}\sum_{i,j=2}^{n}\varrho_{ij}R_{1i1j}-\frac{n^2+3n-58}{120}\varrho^2_{11}\\
&-\frac{7n^2+21n+194}{120}\nabla^2_{11}\varrho_{11}-\frac{(n+1)(n+2)}{36}\sum_{i,j=2}^{n}R^2_{1i1j}\\
&+\frac{n^2+3n+62}{180}\sum_{i=2}^{n}\varrho^2_{1i}-\frac{(n+1)(n+2)}{60}\sum_{i,j,k=2}^{n}R^2_{1ijk}+\frac{n^2-3n+8}{6}\varkappa U\tau\\
&-\frac{n^2+3n+14}{6}\varkappa\nabla_{1}\varrho_{1U}-\frac{n^2+3n+14}{12}\varkappa\nabla_{U}\varrho_{11}\Big\}(\sigma(t)).
\end{aligned}
$$

Here c_{n-2} is the volume of the unit sphere $S^{n-2}(1)$ in E^{n-1}, i.e.

$$c_{n-2}=\frac{(n-1)\,\pi^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!}\quad \textit{where}\quad \left(\frac{n-1}{2}\right)!=\Gamma\left(\frac{n+1}{2}\right).$$

It follows from Theorem 4.5 that $T_\sigma(s)$ does not depend on the torsion of the curve σ but in general the curvature of σ does not disappear. In [9] we showed that the volume $S_\sigma(s)$ of P_s does not depend on the curvature of σ when M is flat or a rank one symmetric space but only depends on the length of σ. For E^n and $S^n(\lambda)$ this was already contained in a remarkable result of H. Weyl [12]. He proved that the volume of a tube about an arbitrary submanifold in E^n and $S^n(\lambda)$ does not depend on the embedding but only depends on the intrinsic geometry of the submanifold. It is shown in [9] that this is still true for curves in the other rank one symmetric spaces but it fails to be true for other dimensional submanifolds (see [10]).

To prove a similar result for the total scalar curvature $T_\sigma(s)$ one may compute explicitely the total scalar curvature. We will do this for $S^n(\lambda)$ in section 6. For the other rank one symmetric spaces the explicit calculations become quickly very complicated. In [10] another method will be developed by considering the *integrated mean curvatures* of a tube P_s. The key argument is that a tube P_{s+r} of radius $s+r$ may be considered as a parallel hypersurface of the tube P_s. So it remains to give a formula for the volume of parallel hypersurfaces. These complete Steiner formulas for E^n and the other rank one symmetric spaces are given in [1]. We obtain

Theorem 5.2. *Let P_s be a tube about a curve σ in* Euclidean *space or in a rank one symmetric space. Then the total scalar curvature $T_\sigma(s)$ of P_s does not depend on the embedding of σ; it depends only on the radius s and the length of σ.*

In what follows we will call E^n and the other rank one symmetric spaces the *model spaces*. Next we will characterize these model spaces locally by means of the total scalar curvature of small tubes about sufficiently short geodesics. The technique is the same as for the similar theorems using the volume $S_\sigma(s)$ [9]. We refer to that paper for more details. Before doing this we first characterize EINSTEIN spaces.

Theorem 5.3. *Let M be an n-dimensional* RIEMANNian *manifold* $(n>3)$ *with the property that for all small s and all sufficiently short geodesics σ, M has the same total scalar curvature functions $T_\sigma(s)$ up to order s^{n-2} as in an n-dimensional* EINSTEIN *manifold M'. Then M itself is an* EINSTEIN *manifold.*

Proof. For M' (5.2) reduces to

$$T_\sigma(s)=c_{n-2}s^{n-2}\int_a^b(n-3)\,(n-2)\left(s^{-2}-\frac{\tau'}{6n}\right)(\sigma(t))\,dt+0(s^n)\,. \tag{5.3}$$

By comparing the expansion (5.1) for M with (5.3) we obtain that

$$\frac{n-3}{n-1}\{(n-4)\,\tau+(n+2)\,\varrho_{11}\}=(n-3)\,(n-2)\,\frac{\tau'}{n} \tag{5.4}$$

for any othonormal basis of M_m. Then (5.4) implies at once that M is an EINSTEIN manifold.

Next we give the other characterization theorems. We will state the theorem in general and prove it for example when the model space is $S^n(\lambda)$. For the other spaces the proof is similar (see [9]).

Theorem 5.4. *Let M be an n-dimensional* RIEMANNian *manifold* $(n>3)$ *with adapted holonomy group (as in the model space) and with the property that for all small s and all sufficiently short geodesics σ, M has the same total scalar curvature functions $T_\sigma(s)$ as in an n-dimensional model space. Then M is locally isometric to that model space.*

Proof. We consider $S^n(\lambda)$ as model space. First Theorem 5.3 implies that M is an EINSTEIN manifold. Using this result it follows from (5.2) that the next condition is

$$B(n)\,(M)=B(n)\,(S^n(\lambda))\,. \tag{5.5}$$

To use this relation in an elegant way we first consider

$$T_m^x(s)=\lim_{L(\sigma)\to 0}\frac{T_\sigma(s)}{L(\sigma)}\,,\quad x=\dot\sigma(0)$$

and then average this over $S^{n-1}(1)$ as x varies on the unit sphere in M_m, i.e.

$$A\,T_m(s)=\frac{1}{c_{n-1}}\int_{S^{n-1}(1)}T_m^x(s)\,dx\,.$$

By using the integration formulas as given in [5], [9] we obtain

$$A T_m(s) = c_{n-2} s^{n-4} \{(n-3)(n-2) + \alpha(n) s^2 + \beta(n) s^4 + 0(s^6)\}(m)$$

where

$$\alpha(n) = -\frac{(n-3)(n-2)}{6n} \tau , \tag{5.6}$$

$$\begin{aligned}\beta(n) = \frac{1}{360(n-1)n(n+2)} \{&5(n+1)(n^2-7n-6)\tau^2 \\ &-6(3n^3-22n^2+33n-2)\Delta\tau + 8(n+1)(n^2+5n+21)\|\varrho\|^2 \\ &-3(n+1)(n+2)(n+3)\|R\|^2\}(m) .\end{aligned} \tag{5.7}$$

So (5.5) implies

$$\beta(n)(M) = \beta(n)\big(S^n(\lambda)\big) \tag{5.8}$$

and from (5.6) we obtain $\tau = \tau'$. Since M and M' are both EINSTEIN spaces we have also that $\|\varrho\|^2 = \|\varrho'\|^2$ and so, from (5.7) and (5.8) we get immediately $\|R\|^2 = \|R'\|^2$. Since $\|R'\|^2 = \frac{2}{n(n-1)} \tau'^2$ we arrive at

$$\|R\|^2 = \frac{2}{n(n-1)} \tau^2$$

which means that M is a space of constant curvature (see for example [5], [9]). Then $\tau = \tau'$ implies at once that the sectional curvature is λ.

In Theorem 5.3 and Theorem 5.4 we consider only the case $n > 3$. The reason for this is the following. It will follow from the formulas in the next section that $T_\sigma(s) = 0$ for E^3, $S^3(\lambda)$ $(\lambda > 0)$ and $H^3(\lambda)$ $(\lambda < 0)$. This follows also from the GAUSS-BONNET theorem. Conversely, suppose that the total scalar curvature $T_\sigma(s)$ for all P_s in a RIEMANNian manifold M with dimension 3 is equal to zero. Then $B(3) = 0$ is equivalent to

$$\nabla^2_{11}\tau = 2\nabla^2_{11}\varrho_{11} . \tag{5.9}$$

This does not enable us to conclude that the space has constant curvature. Indeed we give a counterexample. Let M be a small geodesic sphere in $CP^2(\mu)$. It is proved in [5] that the RICCI tensor of M satisfies $\nabla_X \varrho_{XX} = 0$ for all $X \in \mathfrak{X}(M)$. This implies that τ is constant and also that $\nabla^2_{XX}\varrho_{XX} = 0$ for all $X \in \mathfrak{X}(M)$. Hence (5.9) is satisfied but M is not a space of constant curvature. We do not know if the vanishing of the next term in the power series expansion will make it possible to give an answer.

Note that for a 3-dimensional manifold $\nabla_X \varrho_{XX} = 0$ is equivalent to the condition that $B(3)$ should not depend on the curvature $\varkappa$ of σ.

Finally we want to mention the special behaviour for *4-dimensional manifolds*. These manifolds are characterized by the property

$$\lim_{s \to 0} T_\sigma(s) = \alpha \neq 0 \quad \text{for "one" curve} \quad \sigma ,$$

where α is constant. For all other dimensions we have

$$\lim_{s\to 0} T_\sigma(s)=0 .$$

In particular we prove

Theorem 5.5. *Let M be an n-dimensional manifold such that for all sufficiently small tubes*

$$T_\sigma(s)=\alpha L(\sigma), \quad \alpha \neq 0 \quad \textit{and constant} .$$

Then M is a 4-dimensional locally flat space.

Proof. This follows at once from (5.2).

Further we obtain at once the well-known result:

Corollary 5.6. *Let M be a 4-dimensional locally flat manifold. Then all tubes of sufficiently small radius about curves of the same length have the same total scalar curvature, i.e. $T_\sigma(s)=2c_2L(\sigma)$.*

Theorem 5.7. *An n-dimensional* RIEMANN*ian manifold is a 4-dimensional* EINSTEIN *manifold if and only if*

$$\lim_{s\to 0} T_\sigma(s)=\alpha\neq 0, \quad \lim_{s\to 0} T_\sigma^{(s)-\alpha} s^{-2}=\beta ,$$

α, β constants, for all sufficiently small tubes. Moreover M is RICCI *flat if $\beta=0$.*

6. Some Complete Formulas

For the flat space and the rank one symmetric spaces it is possible to give the *complete* power series expansions. In this section we will give an example. We consider the space $S^n(\lambda)$ with curvature $\lambda>0$. For the negative curvature case it suffices to change all the trigonometric functions into the corresponding hyperbolic functions.

For $S^n(\lambda)$ the JACOBI equation can be solved explicitly. By doing this we obtain for the FERMI vector fields (see [9]):

$$\left.\frac{\partial}{\partial x_1}\right|_{\gamma(s)}=\left\{\cos\sqrt{\lambda}\, s-\frac{\sin\sqrt{\lambda}\, s}{\sqrt{\lambda}}\,\varkappa(m)\, g(U,\gamma')\,(m)\right\} e_1-\frac{\sin\sqrt{\lambda}\, s}{\sqrt{\lambda}}\sum_{\alpha=3}^{n}\perp_{\alpha 2}(m)\, e_\alpha,$$

$$\left.\frac{\partial}{\partial x_2}\right|_{\gamma(s)}=\gamma'(s)=e_2 ,$$

$$\left.\frac{\partial}{\partial x_\alpha}\right|_{\gamma(s)}=\frac{\sin\sqrt{\lambda}\, s}{\sqrt{\lambda}\, s}\, e_\alpha, \quad \alpha=3,\ldots,n .$$

Hence

$$B(s)=\begin{bmatrix} \cos\sqrt{\lambda}s-\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\varkappa(m)\,g(U,\gamma')\,(m) & 0 & 0 & \dots & 0\\ -\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\perp_{32}(m) & \frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}} & 0 & \dots & 0\\ -\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\perp_{42}(m) & 0 & \frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ -\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\perp_{n2}(m) & 0 & 0 & \dots & \frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}} \end{bmatrix}.$$

From this we arrive with (3.5) at

$$C(s)=s\begin{bmatrix} \dfrac{-\varkappa(m)\,g(U,\gamma')\,(m)-\sqrt{\lambda}\tan\sqrt{\lambda}s}{1-\frac{1}{\sqrt{\lambda}}\varkappa(m)\,g(U,\gamma')\,(m)\tan\sqrt{\lambda}s} & 0 & \dots & 0\\ 0 & \sqrt{\lambda}\cot\sqrt{\lambda}s & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & \sqrt{\lambda}\cot\sqrt{\lambda}s \end{bmatrix}.$$

Hence we have from (3.4)

$$S_{11}(s)=\frac{-\varkappa(m)\,g(U,\gamma')\,(m)-\sqrt{\lambda}\tan\sqrt{\lambda}s}{1-\frac{1}{\sqrt{\lambda}}\varkappa(m)\,g(U,\gamma')\,(m)\tan\sqrt{\lambda}s},$$

$$S_{1\alpha}(s)=0$$

$$S_{\alpha\beta}(s)=\sqrt{\lambda}\cot\sqrt{\lambda}s\,\delta_{\alpha\beta},\quad \alpha,\beta=3,\dots,n\,.$$

This means that P_s is a *quasi-umbilical hypersurface* [3]. The converse is also true (see [6]).

Next we have from (3.10) or by explicit calculation from $B(s)$:

$$\omega_{1\dots n}(\exp_m s\gamma'(0))=\left(\cos\sqrt{\lambda}s-\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}}\varkappa(m)\,g(U,\gamma')\,(m)\right)\left(\frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}s}\right)^{n-2}.$$

Further we compute the curvature of P_s. Using the GAUSS equation we obtain

$$R^T_{1\alpha1\beta}(p)=\lambda\left(1-\frac{\varkappa(m)\,g(U,\gamma')\,(m)\cot\sqrt{\lambda}s+\sqrt{\lambda}}{\sqrt{\lambda}-\varkappa(m)\,g(U,\gamma')\,(m)\tan\sqrt{\lambda}s}\right)\delta_{\alpha\beta}\,,$$

$$R^T_{1\alpha\beta\gamma}(p)=0\,,$$

$$R^T_{\alpha\beta\gamma\delta}(p)=\frac{\lambda}{\sin^2\sqrt{\lambda}s}\,(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})\,,$$

$\alpha,\beta,\gamma,\delta=3,\dots,n\,.$

Contraction gives

$$\varrho^T_{11}(p)=(n-2)\,\lambda\left(1-\frac{\varkappa(m)\,g(U,\gamma')\,(m)\cot\sqrt{\lambda}\,s+\sqrt{\lambda}}{\sqrt{\lambda}-\varkappa(m)\,g(U,\gamma')\,(m)\tan\sqrt{\lambda}\,s}\right),$$

$$\varrho^T_{1\alpha}(p)=0\,,$$

$$\varrho^T_{\alpha\beta}(p)=\lambda\left\{\frac{n-3}{\sin^2\sqrt{\lambda}\,s}+1-\frac{\varkappa(m)\,g(U,\gamma')\,(m)\cot\sqrt{\lambda}\,s+\sqrt{\lambda}}{\sqrt{\lambda}-\varkappa(m)\,g(U,\gamma')\,(m)\tan\sqrt{\lambda}\,s}\right\}\delta_{\alpha\beta}\,,$$

$\alpha,\beta=3,\ldots,n$. From this it follows immediately that the tube P_s is η-EINSTEINian.

A second contraction gives the scalar curvature:

$$\tau^T(p)=(n-2)\,\lambda\left\{2+\frac{n-3}{\sin^2\sqrt{\lambda}\,s}-2\,\frac{\varkappa(m)\,g(U,\gamma')\,(m)\cot\sqrt{\lambda}\,s+\sqrt{\lambda}}{\sqrt{\lambda}-\varkappa(m)\,g(U,\gamma')\,(m)\tan\sqrt{\lambda}\,s}\right\}. \tag{6.1}$$

Finally we compute the total scalar curvature. Since

$$\int_{S^{n-2}(1)} g(U,u)\,(m)\,du=0\,,$$

we obtain

$$T_\sigma(s)=c_{n-2}\,(n-2)\,(n-3)\left(\frac{\sin\sqrt{\lambda}\,s}{\sqrt{\lambda}}\right)^{n-4}(\cos\sqrt{\lambda}\,s)\,L(\sigma)\,.$$

This shows that $T_\sigma(s)$ is indeed independent of the embedding. Further we have $T_\sigma(s)=0$ for $n=3$.

For $H^n(\lambda)$ $(\lambda<0)$ we have

$$T_\sigma(s)=c_{n-2}\,(n-2)\,(n-3)\left(\frac{\sinh\sqrt{-\lambda}\,s}{\sqrt{-\lambda}}\right)^{n-4}(\cosh\sqrt{-\lambda}\,s)\,L(\sigma)\,,$$

and for E^n we obtain

$$T_\sigma(s)=c_{n-2}\,(n-2)\,(n-3)\,s^{n-4}L(\sigma)\,.$$

In the same way we can obtain the formulas for $CP^n(\mu)$, $QP^n(\nu)$, Cay $P^2(\zeta)$ and their noncompact duals. However the formulas are much more complicated. We delete them here (see [6]).

Note that τ^T may be computed using the formula

$$\tau^T=\tau-2\varrho_{22}+h^2-\|S\|^2$$

which is derived directly from the GAUSS equation. Here $\|S\|$ denotes the length of the second fundamental form.

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