CURVATURE, TORSION AND THE FRENET FRAME

1. Curvature

In this section we are given a curve

$$s \mapsto \mathbf{R}(s)$$

parametrized by arclength, i.e. the derivative $\mathbf{R}'(s)$ has magnitude 1, it is the unit tangent vector $\mathbf{T}(s)$. The change of $\mathbf{T}(s)$ relative to arclength is a measure for the curvature of the curve.

Definition. The magnitude of $\mathbf{T}'(s)$ is called curvature κ (at the point given by the vector $\mathbf{R}(s)$):

$$\kappa = \kappa(s) = |\mathbf{T}'(s)|. \tag{1.1}$$

We can say something else about the vector $\mathbf{T}'(s)$, namely that it is perpendicular to $\mathbf{T}(s)$. This is because $\mathbf{T}$ is a unit vector, therefore $\mathbf{T}\cdot\mathbf{T} = 1$ and if we differentiate using the product rule and the symmetry of the dot producr we get $\mathbf{T} \cdot \mathbf{T}' = 0$, so $\mathbf{T}'$ is perpendicular to $\mathbf{T}$.

Assuming that $\mathbf{T}'$ is different from the zero vector (that is assuming that κ is different from 0) we can normalize $\mathbf{T}'$ by deviding by its magnitude and get a unit normal vector $\mathbf{N}$:

$$\mathbf{N}(s) = \frac{1}{|\mathbf{T}'(s)|}\mathbf{T}'(s) = \frac{1}{\kappa(s)}\mathbf{T}'(s).$$

Definition: The vector $\mathbf{N}(s)$ is called the *principal normal vector.*

With this definition we have

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s). \tag{1.2}$$

2. Torsion

In this section we still work with parametrization by arclength. We assume that the curvature does not vanish and therefore the unit vectors $\mathbf{T}(s)$ and $\mathbf{N}(s)$ are well defined; recall that they are perpendicular. If we form a vector product of $\mathbf{T}(s)$ and $\mathbf{N}(s)$,

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s),$$

we get a third unit vector which is *perpendicular to both* $\mathbf{T}$ *and* $\mathbf{N}$.

Consider the plane $E(s)$ parallel to $\mathbf{T}(s)$ and $\mathbf{N}(s)$ which goes through the point determined by $\mathbf{R}(s)$. The derivative $\mathbf{B}'(s)$ is a measure on how the curve winds out of this plane.

Let us examine this derivative. By the product rule for vector products we have

$$\mathbf{B}'(s) = \mathbf{T}'(s) \times \mathbf{N}(s) + \mathbf{T}(s) \times \mathbf{N}'(s).$$

However $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ and since $\mathbf{N} \times \mathbf{N} = 0$ the first term drops out. So we get

$$\mathbf{B}'(s) = \mathbf{T}(s) \times \mathbf{N}'(s).$$

This means that $\mathbf{B}'(s)$ is perpendicular to $\mathbf{T}(s)$. But since $\mathbf{B}(s)$ is a unit vector we have that $\mathbf{B}'(s)$ is perpendicular to $\mathbf{B}(s)$ as well. Thus we have seen that $\mathbf{B}'(s)$ is perpendicular to both $\mathbf{B}(s)$ and $\mathbf{T}(s)$ which means that $\mathbf{B}'(s)$ has to be parallel to $\mathbf{N}(s)$. Thus we get

$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s) \tag{2.1}$$

for some scalar function $\tau(s)$.

Definition. The number $\tau(s)$ determined by (2.1) is called the *torsion* of the curve at the point $\mathbf{R}(s)$.

By taking the dot product with $\mathbf{N}(s)$ in (2.1) and using $\mathbf{N} \cdot \mathbf{N} = 1$ we also see that we can define $\tau(s)$ by

$$\tau(s) = -\mathbf{B}'(s) \cdot \mathbf{N}(s).$$

Notice that the magnitude of $\mathbf{B}'(s)$ is equal to the absolute value of the torsion, $|\tau(s)|$, but (2.1) determines also the sign of the torsion.

Example: The helix

$$\mathbf{R}(s) = a\cos(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{i} + a\sin(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{j} + \frac{bs}{\sqrt{a^2+b^2}}\mathbf{k}$$

is parametrized by arclength.

We compute

$$\mathbf{T}(s) = \mathbf{R}'(s) = -\frac{a}{\sqrt{a^2+b^2}}\sin(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{j} + \frac{b}{\sqrt{a^2+b^2}}\mathbf{k}$$

and

$$\mathbf{T}'(s) = \mathbf{R}''(s) = -\frac{a}{a^2+b^2}\cos(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{i} - \frac{a}{a^2+b^2}\sin(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{j}$$

The magnitude of this vector is equal to $\frac{a}{a^2+b^2}$ and thus we get

$$\kappa(s) = \frac{a}{a^2+b^2}$$

and

$$\mathbf{N}(s) = -\cos(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{i} - \sin(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{j}.$$

Now to the torsion and $\mathbf{B}(s)$. Verify that the vector product of $\mathbf{T}$ and $\mathbf{N}$ is given here by

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = \frac{b}{\sqrt{a^2+b^2}}\sin(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{i} - \frac{b}{\sqrt{a^2+b^2}}\cos(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{j} + \frac{a}{\sqrt{a^2+b^2}}\mathbf{k}.$$

Now compute

$$\mathbf{B}'(s) = \frac{b}{a^2+b^2}\cos(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{i} + \frac{b}{a^2+b^2}\sin(\tfrac{s}{\sqrt{a^2+b^2}})\mathbf{j}$$

and thus we have

$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s) \text{ with } \tau(s) = \frac{b}{a^2+b^2}.$$

I.e. the torsion is constant (independent of s) here.

3. **More on the Frenet frame**

In the previous section we got formulas on the derivatives with respect to arclength of **T** and **B**, namely they are parallel to the principal normal vector **N**:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \tag{3.1}$$

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}. \tag{3.2}$$

What can we say about $\frac{d\mathbf{N}}{ds}$? The answer is formula (3.3) below.

Since **T**, **N** and **B** are mutually perpendicular unit vectors we can express

$$\frac{d\mathbf{N}}{ds}(s) = c_1(s)\mathbf{T}(s) + c_2(s)\mathbf{N}(s) + c_3(s)\mathbf{B}(s)$$

with coefficients $c_1(s)$, $c_2(s)$, $c_3(s)$ and it is our job to find these coefficients.

Since **N** is a unit vector we know that $\frac{d\mathbf{N}}{ds}$ must be perpendicular to **N**. We now take a dot product with **N** for both sides of the last displayed formula. We use that **T** and **B** are perpendicular to **N** and $\mathbf{N}\cdot\mathbf{N} = 1$. The outcome is $c_2(s) = 0$. Thus the last display becomes

$$\frac{d\mathbf{N}}{ds}(s) = c_1(s)\mathbf{T}(s) + c_3(s)\mathbf{B}(s).$$

If we take a dot product with **T** and use that **B** and **T** are perpendicular and that $\mathbf{T}\cdot\mathbf{T} = 1$ then we get

$$\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}(s) = c_1(s)\mathbf{T}\cdot\mathbf{T}(s) = c_1(s)$$

But if we differentiate the relation $\mathbf{N}(s)\cdot\mathbf{T}(s) = 0$ we get by the product rule $\frac{d\mathbf{N}}{ds}\cdot\mathbf{T} + \mathbf{N}\cdot\frac{d\mathbf{T}}{ds} = 0$ and hence $\frac{d\mathbf{N}}{ds}\cdot\mathbf{T} = -\mathbf{N}\cdot\frac{d\mathbf{T}}{ds} = -\kappa\mathbf{N}\cdot\mathbf{N} = -\kappa$. Thus

$$c_1(s) = -\kappa(s).$$

To compute $c_2(s)$ we take the dot product $\frac{d\mathbf{N}}{ds}\cdot\mathbf{B}$ and get

$$\frac{d\mathbf{N}}{ds}\cdot\mathbf{B}(s) = c_3(s)$$

and after differentiating $\mathbf{N}\cdot\mathbf{B} = 0$ we get $\frac{d\mathbf{N}}{ds}\cdot\mathbf{B} = -\mathbf{N}\cdot\frac{d\mathbf{B}}{ds} = \mathbf{N}\cdot\tau\mathbf{N} = \tau$. Hence

$$c_3(s) = \tau(s).$$

Thus we get

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s). \tag{3.3}$$

The formulas (3.1), (3.2) and (3.3) are called **Frenet's formulas**. Here they are together:

$$\begin{aligned} \frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N} \end{aligned}$$

4. Computation of Curvature

We now describe a curve by a parametrization

$$t \mapsto \mathbf{r}(t)$$

where $\mathbf{r}' \neq 0$.

The arclength at time t is given by

$$s(t) = \int_0^t |\mathbf{r}'(\tau)| d\tau$$

and the speed is its derivative

$$s'(t) = |\mathbf{r}'(t)|,$$

the magnitude of the velocity vector.

We now wish to derive a formula for the curvature at time t. That means we want a formula for $\kappa(s)$ when $s = s(t)$. We set $\widetilde{\kappa}(t) = \kappa(s(t))$ and we are aiming for a formula for $\widetilde{\kappa}(t)$.

Before we begin we need some more notation. If we reparametrize the curve by arclength $s \mapsto \mathbf{R}(s)$, then the two parametrizations are linked by

$$\mathbf{r}(t) = \mathbf{R}(s(t))$$

Consider the unit tangent vector $\mathbf{T}$, then this is a function of s. If you want to express it as a function of time then you have to plug in $s(t)$ for s and thus you get a new function

$$\widetilde{\mathbf{T}}(t) = \mathbf{T}(s(t))$$

which is the unit tangent vector at time t (corresponding to the s with $s = s(t)$. Similarly (assuming nonzero curvature) you can express the principal unit normal vector $\mathbf{N}(s)$ as a function of t by plugging in $s(t)$ so we get a new function

$$\widetilde{\mathbf{N}}(s) = \mathbf{N}(s(t)).$$

Now the velocity vector is the unit tangent vector times speed, i.e.

$$\mathbf{r}'(t) = s'(t)\mathbf{T}(s(t)). \tag{4.1}$$

We differentiate with repect to t (using the product rule) and get

$$\mathbf{r}''(t) = s''(t)\mathbf{T}(s(t)) + s'(t)\frac{d}{dt}\mathbf{T}(s(t)).$$

By the chain rule $\frac{d}{dt}\mathbf{T}(s(t)) = \mathbf{T}'(s(t))s'(t)$ (where $\mathbf{T}'(s(t))$ means that we differentiate with respect to s and then plug in $s(t)$). We also know that $\mathbf{T}' = \kappa\mathbf{N}$ as functions of s. That means we get from the last displayed formula

$$\mathbf{r}''(t) = s''(t)\mathbf{T}(s(t)) + (s'(t))^2\kappa(s(t))\mathbf{N}(s(t)). \tag{4.2}$$

Now comes a *trick* which exploits a certain cancellation in an effective way. It turns out that we get a nice formula if we consider the vector product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ and use the above formulas. We get

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= s'(t)\mathbf{T}(s(t)) \times \Big(s''(t)\mathbf{T}(s(t)) + (s'(t))^2\kappa(s(t))\mathbf{N}(s(t))\Big)\\ &= s'(t)s''(t)\mathbf{T} \times \mathbf{T}(s(t)) + s'(t)^3\kappa(s(t))\mathbf{T}(s(t)) \times \mathbf{N}(s(t))\end{aligned}$$

but $\mathbf{T} \times \mathbf{T} = 0$. Also by definition of $\mathbf{B}$ we have $\mathbf{T} \times \mathbf{N} = \mathbf{B}$. Thus

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = s'(t)^3\kappa(s(t))\mathbf{B}(s(t)).$$

Now we consider the magnitudes and use $s'(t) = |\mathbf{r}'(t)|$ to get

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\mathbf{r}'(t)|^3\kappa(s(t)).$$

(Recall that κ was defined to be nonnegative). The last formula gives a theorem for the computation of the curvature as a function of t.

Theorem. *The curvature as a function of T is given by*

$$\widetilde{\kappa}(t) = \kappa(s(t)) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}. \tag{4.3}$$

This is the vector formula for curvature on p. 947 in the book, it is effective for computation (there the "velocity" $\mathbf{r}'$ is denoted by $\mathbf{v}$ and the "acceleration" $\mathbf{r}''$ is denoted by $\mathbf{a}$).

5. **Computation of Torsion**

We continue the same notation as in the previous section. We write $\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}(t) := \mathbf{T}(s(t))$ etc.

We shall get the following formula for the torsion which I state in the beginning.

Theorem. *If the curve has nonvanishing curvature then the torsion (as a function of the parameter t) is given by*

$$\widetilde{\tau}(t) = \tau(s(t)) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}. \tag{5.1}$$

We first recall

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = s'(t)^3\widetilde{\kappa}(t)\widetilde{\mathbf{B}}(t). \tag{5.2}$$

and deduce also

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2 = s'(t)^6 \widetilde{\kappa}(t)^2. \tag{5.3}$$

We differentiate the formula (4.2) and obtain

$$\begin{aligned}\mathbf{r}'''(t) &= \frac{d}{dt}\Big[s''\widetilde{\mathbf{T}} + (s')^2\widetilde{\kappa}\widetilde{\mathbf{N}}\Big] \\ &= s'''\widetilde{\mathbf{T}} + s''\widetilde{\mathbf{T}}' + ({s'}^2\widetilde{\kappa})'\widetilde{\mathbf{N}} + (s')^2\widetilde{\kappa}\widetilde{\mathbf{N}}'.\end{aligned}$$

We are interested in the component of $\mathbf{r}'''$ with respect to the binormal vector $\mathbf{B}$ and since $\widetilde{\mathbf{B}} \cdot \widetilde{\mathbf{T}} = 0$, $\widetilde{\mathbf{B}} \cdot \widetilde{\mathbf{N}} = 0$ and $\mathbf{B} \cdot \widetilde{\mathbf{T}}' = \widetilde{\kappa} s' \mathbf{B} \cdot \widetilde{\mathbf{N}} = 0$ we see that this component is is

$$\mathbf{r}'''(t) \cdot \widetilde{\mathbf{B}}(t) = (s'(t))^2 \widetilde{\kappa}(t) \widetilde{\mathbf{N}}'(t) \cdot \widetilde{\mathbf{B}}(t).$$

Now $\widetilde{\mathbf{N}}'(t) = \frac{d}{dt}\widetilde{\mathbf{N}}(t) = \frac{d}{dt}\mathbf{N}(s(t)) = \frac{d\mathbf{N}}{ds}(s(t))s'(t)$ and by (3.3) we have $\frac{d\mathbf{N}}{ds}(s(t)) = -\widetilde{\kappa}(t)\widetilde{\mathbf{T}}(t) + \widetilde{\tau}(t)\widetilde{\mathbf{B}}(t)$. Thus

$$\begin{aligned}\mathbf{r}'''(t) \cdot \widetilde{\mathbf{B}}(t) &= (s'(t))^3 \widetilde{\kappa}(t)(-\widetilde{\kappa}(t)\widetilde{\mathbf{T}}(t) \cdot \widetilde{\mathbf{B}}(t) + \widetilde{\tau}(t)\widetilde{\mathbf{B}}(t) \cdot \widetilde{\mathbf{B}}(t)) \\ &= (s'(t))^3 \widetilde{\kappa}(t) \widetilde{\tau}(t). \qquad (5.4)\end{aligned}$$

Going back to (5.2) we get

$$\begin{aligned}&(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = s'(t)^3 \widetilde{\kappa}(t) \widetilde{\mathbf{B}}(t) \cdot \mathbf{r}'''(t) \\ &= s'(t)^6 \widetilde{\kappa}(t)^2 \widetilde{\tau}(t)\end{aligned}$$

and from (5.3) we also get

$$(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = |\mathbf{r}'(t) \times \mathbf{r}''(t)|^2 \widetilde{\tau}(t)$$

which yields the asserted formula in the theorem. □

Remark. The expression $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$ can also be written as a three by three determinant (see p. 949 in the textbook). The formula there follows from the above formula by using the determinant formula for the triple scalar product (p. 878).

Exercise. Consider

$$\mathbf{r}(t) = t\mathbf{i} + a\frac{t^2}{2}\mathbf{j} + b\frac{t^3}{6}\mathbf{k}$$

where a and b are given constants (not both 0). Compute the unit tangent vector, principal unit normal vector, binormal vector, (whenever these quantities make sense). Compute the curvature and torsion at time t (and point $\mathbf{r}(t)$).